B- and D- Meson Decay Constants in QCD

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The decay amplitude:

\[ A(B^- \to \tau^- \bar{\nu}_\tau)_{SM} = \frac{G_F}{\sqrt{2}} V_{ub} \left\langle 0 \left| \bar{u} \gamma_\mu \gamma_5 b \right| B \right\rangle \bar{\tau} \gamma^\mu (1 - \gamma_5) \nu_\tau \]

- Hadronic matrix element \( \Rightarrow \) decay constant

\[ \left\langle 0 \left| \bar{u} \gamma^\mu \gamma_5 b \right| B(p_B) \right\rangle = i p_B^\mu f_B, \quad p_B^2 = m_B^2 \]

- Partial width: (suppressed for \( \ell = \mu, e \))

\[ BR(B^- \to \tau^- \bar{\nu}_\tau)_{SM} = \frac{G_F^2 |V_{ub}|^2}{8\pi} m_\tau^2 m_B \left(1 - \frac{m_\tau^2}{m_B^2}\right)^3 f_B^2 \tau_B^- \]

\( \{ b \to u \) flavour-changing transition\} \( \otimes \) \{QCD colour forces\)

**B \to \tau \nu_\tau \) decay in Standard Model**
Flavour-changing transitions in $P \to \ell \nu_\ell$

- flavour structure of Standard Model: CKM matrix

$$j_W^\mu = (\bar{u} \, \bar{c} \, \bar{t}) \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \gamma^\mu (1 - \gamma_5) \begin{pmatrix} d \\ s \\ b \end{pmatrix},$$

- various $P \to \ell \bar{\nu}_\ell$, $P$ -pseudoscalar meson, $J^P = 0^-$

<table>
<thead>
<tr>
<th>meson $P$</th>
<th>$\pi^-$</th>
<th>$K^-$</th>
<th>$D^-$</th>
<th>$D_s^-$</th>
<th>$B^-$</th>
</tr>
</thead>
<tbody>
<tr>
<td>flavour content</td>
<td>$</td>
<td>\bar{u}d\rangle$</td>
<td>$</td>
<td>\bar{u}s\rangle$</td>
<td>$</td>
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<tr>
<td>$V_{CKM}$ element</td>
<td>$V_{ud}$</td>
<td>$V_{us}$</td>
<td>$V_{cd}$</td>
<td>$V_{cs}$</td>
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<tr>
<td>decay constant</td>
<td>$f_\pi$</td>
<td>$f_K$</td>
<td>$f_D$</td>
<td>$f_{D_s}$</td>
<td>$f_B$</td>
</tr>
</tbody>
</table>

$B_c \to \ell \nu_\ell$ not accesible

- new physics can contribute: $H^\pm,..., B \to \tau \nu_\tau$ most sensitive
Rare leptonic decays: $B_{s,d} \to \ell^+ \ell^-$

- in SM $t$, $W$, $Z$-loops, sensitive to $V_{ts}V_{tb}^*$,
- more realistic chances to find/constrain new physics
- after integrating out heavy loops: the hadronic matrix element in decay amplitude reduced to
  \[ \langle 0 | \bar{s} \gamma^\mu \gamma_5 b | B_s(p_B) \rangle = i p_B^{\mu} f_{B_s}, \text{ or } f_{B_d} \]
  \[ f_{B_d} \simeq f_{B_u} \equiv f_B \text{ (isospin symmetry),} \]
  \[ f_{B_s} \neq f_B, \text{ (SU(3)$_f$ violation)} \]
Is it possible to calculate $f_B$?

- Quark model of $B$-meson: a bound state of heavy $b$ and light $\bar{u}$

- $B \rightarrow$ vacuum transition matrix element
  \[
  \langle 0 | \bar{u} \gamma^\mu \gamma_5 b | B(p_B) \rangle = i p_B^\mu f_B
  \]

- $\bar{b} - u$ interaction potential?
  (e.g., one-gluon exchange, similar to $p - e$ potential in H-atom)
  $\Rightarrow$ $B$-meson wave function $\psi_B(x)$, $f_B \sim \psi(0)$

- Two important energy/mass scales:
  $m_b \sim 4.6 - 4.8$ GeV ("pole" mass),
  $\bar{\Lambda} \sim m_B - m_b \sim 500-700$ MeV,
  $m_u \ll \bar{\Lambda}$, light (anti)quark relativistic,
  no $b\bar{u}$ "interaction potential" and/or $B$-meson "wave-function"!
\( \bar{\Lambda} \sim m_B - m_b \sim 500-700 \text{ MeV}, \) 
the energy scale of colour forces binding \( b \) and \( \bar{u} \) inside the \( B \)

- \( \alpha_s(\bar{\Lambda}) \) too large for a perturbative expansion
- no expansion in \( \alpha_s \) can be used: nonperturbative QCD
$|B^{-}\rangle = |b\bar{u} \oplus \text{gluons} \oplus \text{soft quark-antiquark pairs} \rangle$

\[ \langle 0 |\ , \text{the QCD vacuum, (the lowest energy state, no hadrons) populated by fluctuating quark-antiquark and gluon fields} \]

\[ \langle 0 |\bar{q}q|0 \rangle \neq 0, \ (q = u, d, s), \]

- spontaneous breaking of chiral symmetry in QCD

[Y. Nambu, Nobel Prize 2008],

\[ \langle 0 |G_{\mu\nu}G^{\mu\nu}|0 \rangle \neq 0, \ ... \ (\text{vacuum condensates}) \]
\(B\)-meson in QCD

- \(m_B\), \(f_B\) and many other \(B\)-meson observables, are determined by long-distance colour forces in QCD.
- No perturbation theory applicable: \(\alpha_s\) too large.
- In addition to "valence" quarks: soft quark-antiquark and gluons are present in \(B\) (related to QCD vacuum effects).
- No \(b - \bar{u}\) interaction potential can be systematically derived.
- No QCD-related "wave function" of quark constituents in \(B\).

A different approach has been developed...
Correlation function of $\bar{u}b$ currents

- formal definition of the vacuum correlation function:

$$\Pi_{\mu\nu}(q^2) = \int d^4x \ e^{iqx} \langle 0| T\{j_\mu^W(x)j_\nu^{W\dagger}(0)\}|0\rangle,$$

a quantum amplitude of emission and absorption of $\bar{u}b$ pair in vacuum by the external current:

$$j_\mu^W = \bar{u}\gamma_\mu\gamma_5 b,$$
$$j_\mu^{W\dagger} = \bar{b}\gamma_\mu\gamma_5 u$$

$$\Pi_{\mu\nu}(q^2) =$$

- the flavour and $J^P$ of the current can vary currents with other meson quantum numbers ($B_s$, $D$, $D_s$, $\pi$, $\rho$, ...),

(any Lorentz-covariant and colour-invariant local operator)
Correlation function far below the $B$ threshold

\[ q \rightarrow b \rightarrow jW \rightarrow jW \rightarrow \pi \]

- 4-momentum of the $b\bar{u}$ pair: \( q = (q_0, \vec{q}) \), \( q^2 = q_0^2 - \vec{q}^2 \), rest frame: \( \vec{q} = 0 \), \( q^2 = q_0^2 \), fix the energy \( q_0 \ll m_B \)

- the $b\bar{u}$-pair is virtual: \( \Delta E \Delta t \sim 1 \), the energy deficit \( \Delta E \sim m_B \), \( \Delta t \sim 1/m_B \)

\( m_B \gg \Lambda_{QCD} \): \( \Delta t \ll 1/\Lambda_{QCD} \)

- virtual quarks propagate during short times, are asymptotically free,

- at \( q^2 \ll m_b^2, m_B^2 \)

\[ \Pi_{\mu\nu}(q^2) \simeq \text{simple loop diagram} \]

\[ \oplus \{ \text{calculable QCD corrections} \} \leftarrow \text{to be added} \]
Correlation function above $B$ threshold

- Hypothetical neutrino-electron scattering, varying c.m. energy $\sqrt{s} = \sqrt{q^2}$,
- $\Pi_{\mu\nu}(q^2)$ is the part of the scattering amplitude

\[ \sqrt{s} \sim 0 \]

\[ \sqrt{s} = m_B \]

\[ \sqrt{s} > m_B \]

\[ \sqrt{s} \gg m_B \]

highly virtual quark pair,

$B$-meson, resonance

excited $B$ mesons

multiple hadrons (continuum)
Hadronic representation of $\Pi_{\mu\nu}(q^2)$

- $\Pi_{\mu\nu}(q^2)$ at $q^2 \ll m_b^2$:
  a short-distance short-lived $b\bar{u}$ -fluctuation,
  $\simeq$ loop diagram

- $\Pi_{\mu\nu}(q^2)$ at $q^2 \geq m_B^2$:
  propagation of $B$ meson and excited $B$ states
  (infinite sum over resonant and multiparticle states)

- the hadronic representation (dispersion relation):

\[
\Pi_{\mu\nu}(q^2) = \frac{\langle 0 | j^W_\mu | B \rangle \langle B | j^W_\nu^\dagger | 0 \rangle}{m_B^2 - q^2} + \sum_{B_{\text{exc}}} \frac{\langle 0 | j^W_\mu | B_{\text{exc}} \rangle \langle B_{\text{exc}} | j^W_\nu^\dagger | 0 \rangle}{m_{B_{\text{exc}}}^2 - q^2}
\]

- rigorous theory derivation:
  analyticity of $\Pi_{\mu\nu}(q^2)$ $\oplus$ Cauchy theorem $\oplus$ unitarity
  (valid in any local quantum-field theory)
a numerical simulation of QCD
on a 3⊕1-dimensional lattice

"We want to find answers for properties of QCD in the non-perturbative domain, where
the usual power series expansion in the coupling constant fails completely. This is
done through a Monte-Carlo evaluation of the Euclidean path integral after a
discretization of space-time on a lattice with spacing \( a \) in all 3+1 dimensions."

R. Sommer, review on lattice QCD
Calculating $f_B$ on the lattice

*(very schematical outline)*

- the correlation function, (3-dim. integration, $\bar{q} \to 0$)
  \[
  C(x_0) = \int d^3x \langle 0| j^W_\mu(x_0, x) j^{W\dagger}_\mu(0)|0 \rangle
  \]
  \[
  j^W_\mu = \bar{q}\gamma_\mu\gamma_5 b, \quad q = u, d,
  \]
- transition to Euclidean time, $x_0 \to i\tau$,
- the sum over hadronic states (at rest):
  \[
  C(\tau) \sim \sum_{B_n=B,B_{exc\ldots}} |\langle 0| j_5|B_n(\vec{p} = 0)\rangle|^2 \frac{\exp(-M_{B_n}\tau)}{2M_{B_n}}
  \]
  \[
  \sim f_B^2 \frac{\exp(-m_{B\tau})}{2m_B} \left[1 + O(\exp(-(m_{B'} - m_B)\tau))\right]
  \]
- calculating $C(\tau)$ at $\tau \to \infty$
  \[
  \Rightarrow \text{fit the } B\text{-meson mass and } f_B
  \]
- replace $b \to c$ and calculate $f_D$, ...
Calculating $f_B$ on the lattice

- $C(\tau)$ in QCD reduced to Euclidean functional integral, incorporates nonperturbative quark-gluon dynamics.
- Discretization: space-time $\rightarrow 3 \oplus 1$-dim. lattice: spacing $a$, lattice size $L$,

$$\int d^3x \rightarrow a^3 \sum \vec{x}$$

- Numerical MC-calculation of the multidimensional integral.
- The answer in units of $a$, normalization to a measured hadronic quantity needed.
- Unquenched $n_f = 2 + 1$ is nowadays standard (light quark-antiquark loops in the lattice action),
Calculating $f_B$ on the lattice

- different methods to describe the quark fields on the lattice
- currently accessible lattices:
  - the physical $b$ quark too heavy: $a \sim 1/m_b$
  - the pion too light: $L \leq 1/m_\pi$
- extrapolations needed
  (use of effective theories, such as HQET, ChPT),
- most recent example: [HPQCD Collab., 008.45621[hep-lat]]
  $n_f = 2 + 1$, $a = 0.09$ fm, $L^3 \times N_T = 24^3 \times 64$
Correlation function: the hadronic representation (dispersion relation):

\[ \Pi_{\mu\nu}(q^2) = \frac{\langle 0|j^W_\mu|B\rangle\langle B|j^W_{\nu\dagger}|0\rangle}{m_B^2 - q^2} + \sum_{B_{exc}} \frac{\langle 0|j^W_\mu|B_{exc}\rangle\langle B_{exc}|j^W_{\nu\dagger}|0\rangle}{m_{B_{exc}}^2 - q^2} \]

\{ l.h.s. in QCD at \( q^2 \ll m_B^2 \} = \{ \text{hadronic sum} \},

(global) quark-hadron duality
ABC of the QCD sum rule method

[M. Shifman, A. Vainshtein, V. Zakharov (1979)]

A. calculate $\Pi_{\mu\nu}(q^2)$ at $q^2 \ll m_b^2$ analytically in a form of loop diagram $\oplus$ QCD corrections

B. match to the hadronic sum (dispersion relation)

C. use (semilocal) quark-hadron duality to estimate the sum over excited states
Transforming to pseudoscalar currents

- simplifying trick: multiply the correlation function,
\[ q^\mu q^\nu \Pi_{\mu\nu}(q^2) \equiv \Pi_5(q^2), \text{ scalar function of } q^2 \]

note that
\[ q^\mu \bar{u} \gamma_\mu \gamma_5 b = (p^b_\mu + p^u_\mu) \bar{u} \gamma_\mu \gamma_5 b = (m_b + m_u) \bar{u} i \gamma_5 b \equiv j_5 \]
(apply Dirac equation for both quark fields)

\[ \Rightarrow \Pi_5(q^2) = \int d^4x \ e^{iqx} \langle 0 | T\{j_5(x)j_5(0)\} | 0 \rangle , \]

- equivalent definition of the decay constant:
  at \( q = p_B \) \( (q^2 = m_B^2) \),
\[ q_\mu \langle 0 | \bar{u} \gamma_\mu \gamma_5 b | B(p_B) \rangle = \langle 0 | j_5 | B(p_B) \rangle = m_B^2 f_B \]
Calculating the correlation function

Adding perturbative gluon effects $\alpha_s(m_B) \ll 1$

Including nonperturbative effects due to condensates

Result: analytical expression for $\Pi_5(q^2)$ in terms of $m_b, m_u$ and universal QCD parameters $\alpha_s, \langle \bar{q}q \rangle, ...$

$b \rightarrow c, u \rightarrow s$, access to $D$ or $B_s$ decay constants
Determination of quark masses

- heavy quark masses in $\overline{\text{MS}}$ (QCD SR for quarkonia)
  \[
  \bar{m}_b(\bar{m}_b) = (4.16 \pm 0.05) \text{GeV}, \\
  \bar{m}_c(\bar{m}_c) = (1.28 \pm 0.03) \text{GeV},
  \]
  [K. Chetyrkin et al. (2009)];

- light quark masses: (QCD SR for strange meson channels)
  \[
  m_s(\mu = 2 \text{GeV}) = (98 \pm 16) \text{MeV}, \\
  m_s \oplus \text{ChPT} \rightarrow m_{u,d} \rightarrow \langle \bar{q}q \rangle
  \]
  [A.K., K. Chetyrkin (2005); M. Jamin, Oller, A. Pich (2006)]

- current accuracy of $\Pi_5(q^2)$ at $q^2 \ll m_b^2$: 
  vacuum condensates with $d \leq 6$
  $O(\alpha_s^2)$ corrections to the loop
  [K. Chetyrkin, M. Steinhauser (2001)]
Quark-hadron duality

\[
\Pi_5(q^2) = \frac{\langle 0 | j_5 | B \rangle \langle B | j_5^\dagger | 0 \rangle}{m_B^2 - q^2} + \sum_{B_{\text{exc}}} \frac{\langle 0 | j_5 | B_{\text{exc}} \rangle \langle B_{\text{exc}} | j_5^\dagger | 0 \rangle}{m_{B_{\text{exc}}}^2 - q^2}
\]

\[
= \frac{1}{\pi} \int_{(m_b+m_u)^2}^{s_0} ds \frac{\text{Im}\Pi_5(s)}{s-q^2} + \frac{1}{\pi} \int_{s_0}^{\infty} ds \frac{\text{Im}\Pi_5(s)}{s-q^2}
\]

- the sum of \(B_{\text{exc}}\)-states is approximated by the calculable integral over \(\text{Im}\Pi_5(s) \Rightarrow s_0\), the effective threshold
- a systematic uncertainty is introduced
- to fix \(s_0\), e.g., calculate \(m_B\) from the same relation

A formula for \(f_B^2\) is obtained (QCD sum rule)

skipping some technical details (Borel transformation)
Deriving the upper bound

• The correlation function $\Pi_5(q^2)$ is a positive definite and real quantity at $q^2 \ll m_b^2$
• the separate $B$ and all $B_{exc}$ contributions are $\geq 0$
• derive an upper bound for $f_B^2$
• use Borel transformation

$$\frac{1}{(m_B^2 - q^2)} \to \exp(-m_B^2/M^2)$$

$$f_B^2 m_B^4 e^{-m_B^2/M^2} + \ldots = \Pi(M^2; m_b, m_u, \alpha_s, \text{cond.}, \mu, \ldots)$$

$$\Rightarrow f_B < \sqrt{\Pi(M^2)/(m_B^4 e^{-m_B^2/M^2})}$$

• no quark-hadron duality assumption involved! relies only on the accuracy of QCD calculation
• the upper bound for $f_D$ and $f_{D_s}$ A.K. PRD 79, 031503(R) (2009)
### $f_B$, current status

<table>
<thead>
<tr>
<th>method</th>
<th>$f_B$ [MeV]</th>
<th>ref.</th>
</tr>
</thead>
<tbody>
<tr>
<td>lattice $^a)$</td>
<td>197 ± 13</td>
<td>HPQCD (Lattice’09+corr.’10)</td>
</tr>
<tr>
<td></td>
<td>212 ± 6 ± 6</td>
<td>FNAL/MILC (Lattice’10)</td>
</tr>
<tr>
<td>QCD SR</td>
<td>210 ± 19</td>
<td>Jamin-Lange ’01</td>
</tr>
<tr>
<td></td>
<td>206 ± 20</td>
<td>Penin-Steinhauser’01</td>
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<tr>
<td></td>
<td>215 ± ...</td>
<td>QCD SR [from JL’01] (prelim.)</td>
</tr>
<tr>
<td>OPE upper bound</td>
<td>270</td>
<td>recalc. bound for $f_{D(s)}$ (prelim.)</td>
</tr>
<tr>
<td>&quot;exp.&quot;</td>
<td>(248 ± 23$<em>{exp}$ ± 25$</em>{V_{ub}}$)</td>
<td>BABAR⊕ Belle, ICHEP10’</td>
</tr>
</tbody>
</table>

"experimental" $f_B$, from:

- average BABAR ⊕ Belle: $BR(B \to \tau \nu_{\tau}) = [1.68 \pm 0.31] \times 10^{-4}$
  [K.Trabelsi, ICHEP’10, Paris]

- $|V_{ub}| = \left(3.92 \pm 0.09|_{exp} ± 0.45|_{th}\right) \times 10^{-3}$
  average of inclusive and exclusive semileptonic $b \to u$
  [CKM fitter group, talk at ICHEP’10]
a tension with QCD predictions increases if one takes $|V_{ub}|$ only from $B \rightarrow \pi l \nu_l$,

the tension is between the ratio of two QCD predictions ($V_{ub}$ cancels) and exp.

possibility of new physics in $b \rightarrow u$ transitions ?

please, be patient, having in mind the experience with $f_{D_s}$, where tension was dramatic in 2008
Status of $f_D$ and $f_{Ds}$

[talk by Bo Xin at CKM-2010, Warwick (Sept. 2010)]

Conclusions (4)

- Before Aug 24, 2010
  In the past two years, there has been a tension between theory and experiment, which also led to much speculation about the existence of New Physics.

- On Aug 24, 2010
  HPQCD submitted their new results, which is 1σ from the HFAG average.

- Also on Aug 24, 2010
  BABAR submitted new fDs results using their full data set. And these supersedes their 2007 results which was an relative measurement.

- Expect an HFAG average soon

BESIII plans to take $20\text{fb}^{-1}$ each at 3770 and 4170 MeV 1-2 % uncertainties on $f_D$ and $f_{Ds}$ expected!

09/08/2010
Leptonic D and Ds Decays
Bo Xin

C. T. H. Davies et al. [HPQCD Collaboration], arXiv: 1008.4018

260.1(5.4)
248.0(2.5)

Expect an HFAG average soon
(My average)
### $D$ and $D_{(s)}$ decay constants, QCD vs experiment

<table>
<thead>
<tr>
<th>method</th>
<th>$f_D$ [MeV]</th>
<th>$f_{Ds}$ [MeV]</th>
<th>ref</th>
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<tbody>
<tr>
<td>lattice$^a)$</td>
<td>$207 \pm 4$</td>
<td>$241 \pm 3$</td>
<td>HPQCD, UKQCD ’08</td>
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<td>$217 \pm 10$</td>
<td>$248 \pm 2.5$</td>
<td>HPQCD ’10</td>
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<td>$260 \pm 10$</td>
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<td>-</td>
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<td>$235 \pm 24$</td>
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<td>exp.</td>
<td>$206.7 \pm 8.9$</td>
<td>$259.0 \pm 6.9$</td>
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<td>$275 \pm 20$</td>
<td>Belle</td>
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<td>$258.6 \pm 9.9$</td>
<td>BABAR</td>
</tr>
<tr>
<td>$\oplus$ CKM</td>
<td>$</td>
<td>V_{cd}</td>
<td>=</td>
</tr>
</tbody>
</table>

- $f_D$, lattice QCD and SR, bound agree with each other and exp.
- $f_{Ds}$, tension of QCD vs exp. is not as dramatic anymore as in 2008
Summary

- **2-point correlation functions** of quark currents allow to relate QCD with hadronic observables, e.g. $f_B$ or $f_D$
- Numerical simulation of correlation function on the lattice with continuous progress, decreasing the errors is the main issue.
- Future goal of lattice calculations $\sim 1 \div 2\%$ accuracy for all $f_B$’s
  - [App.A SuperB report ’07]
- QCD sum rules: analytical calculation in terms of diagrams, duality approximation for excited states $\sim 10\%$ accuracy is probably the limit.
- Future goal: to better assess OPE/input/duality uncertainties.
- A tension in $f_B$ deserves further attention, tension in $f_{Ds}$ seems to be less significant in 2010, than in 2008.
lattice QCD and heavy quarks, introduction for non-experts:
R. Sommer, Les Houches lectures, 2009, 1008.0710 [hep-lat]
R. Sommer 0906.3790 [hep-ph]

QCD sum rules, Introductionary reviews: